

**Temperature correction to Casimir-Lifshitz free energy at low temperatures: Semiconductors**Simen A. Ellingsen<sup>\*</sup> and Iver Brevik<sup>†</sup>*Department of Energy and Process Engineering, Norwegian University of Science and Technology, N-7491 Trondheim, Norway*Johan S. Høye<sup>‡</sup>*Department of Physics, Norwegian University of Science and Technology, N-7491 Trondheim, Norway*Kimball A. Milton<sup>§</sup>*Oklahoma Center for High Energy Physics and Department of Physics and Astronomy, The University of Oklahoma, Norman, Oklahoma 73019, USA*

(Received 20 May 2008; published 15 August 2008)

The Casimir force and free energy at low temperatures have been the subject of focus for some time. We calculate the temperature correction to the Casimir-Lifshitz free energy between two parallel plates made of dielectric material possessing a constant conductivity at low temperatures, described through a Drude-type dielectric function. For the transverse magnetic (TM) mode such a calculation is made. A further calculation for the case of the TE mode is thereafter presented which extends and generalizes previous work for metals. A numerical study is undertaken to verify the correctness of the analytic results.

DOI: [10.1103/PhysRevE.78.021117](https://doi.org/10.1103/PhysRevE.78.021117)

PACS number(s): 72.20.-i, 11.10.Wx, 42.50.Lc, 78.20.Ci

**INTRODUCTION**

There has been an explosion of interest in the Casimir effect [1], generalized to dielectrics by Lifshitz [2], since the modern experiments began with Lamoreaux in 1997 [3]. The zero-temperature Casimir-Lifshitz theory seems to have been confirmed to 1% accuracy over a range from 100 nm to a micrometer [4–12].

However, there has been a continuing controversy over the temperature dependence of this effect. The prescription given in Ref. [13] was seriously questioned by Boström and Sernelius [14] who pointed out that necessarily the transverse electric reflection coefficient at zero frequency must vanish for metals. This discontinuity predicted a linear temperature term at low temperatures, resulting in about a 15% correction to the result found by Lamoreaux. Lamoreaux believes that his experiment could not be in error to this extent [15]. More heatedly, Mostepanenko and collaborators have insisted that this behavior is inconsistent with thermodynamics (the Nernst heat theorem) because it would predict, for an ideal metal, that the free energy has a linear temperature term at low temperature, and hence that the entropy would not vanish at zero temperature [16]. Moreover, they assert that the precision Purdue experiments rule out the linear temperature term in the low-temperature expansion [12].

The issue is as yet unresolved, and is summarized in recent reviews [17,18]. We will not add further to the discussion of this controversy here. Rather, the purpose of this paper is to examine another purported temperature anomaly. In several recent papers [19–22] Geyer, Klimchitskaya, and Mostepanenko have claimed that in real dielectrics, which

possess a very small, but nonzero conductivity which vanishes at  $T=0$ , a similar discontinuity in the transverse magnetic reflection coefficient occurs, which would lead to a similar violation of the Nernst theorem. The same applies to semiconductors whose conductivity vanishes as temperature drops to zero. The solution according to these authors, as in the TE case for good conductors, is to prescribe the effect away. We argue, however, that such a solution is physically unsatisfactory.

In Sec. I, we will review and clarify their argument for a standard Drude-type permittivity model for a weakly conducting material. We thereafter work out the leading-order temperature corrections to the free energy in the cases where the media are assumed to have a finite but small residual conductivity at  $T=0$ , as is implied when a Drude model is employed for taking the conductivity into account. This is a new result to the best of our knowledge (a similar calculation for materials with zero conductivity was undertaken in Ref. [19]). While this calculation does not solve the thermal anomaly brought forth in Refs. [19–21] and reviewed in Sec. I, it serves to further illuminate the mathematical behavior of the free energy of poor conductors at very low temperatures when different models for the dielectric response of the materials are employed. A similar calculation is subsequently performed for the TE mode, which extends that of Ref. [23] in several ways: We allow for the conductivity to be small; we work out one further order of the temperature correction to the free energy; and we allow, for generality, the permittivity to have a finite dielectric constant term in addition to the Drude-type dielectric response due to free charges.

A word about units. For our theoretical calculations, it is most convenient to use Gaussian electromagnetic units, as well as natural space-time units:  $\hbar=c=k_B=1$ . However, for final results, which could be experimentally observed, we use Système International (International System of Units) (SI) units. The mapping between units is very simply carried out by dimensional considerations, using the unit conversion factor  $\hbar c=1.97 \times 10^{-5}$  eV cm. The conductivity transforma-

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tion between Gaussian and SI units is the simple replacement  $4\pi\sigma = \sigma^{\text{SI}}/\epsilon_0$ , where  $\epsilon_0 = 8.85 \times 10^{-12}$  F/m is the absolute permittivity of the vacuum and the notation  $\sigma^{\text{SI}}$  is used to explicate that SI units are used.

### I. TEMPERATURE ANOMALY FOR SEMICONDUCTORS

Here is a simple way to understand the argument of Ref. [19]. Suppose we model a dielectric with some small conductivity by the permittivity function

$$\varepsilon(i\zeta) = 1 + \frac{\bar{\varepsilon} - 1}{1 + \zeta^2/\omega_0^2} + \frac{4\pi\sigma}{\zeta}. \quad (1.1)$$

The essential point is that as  $\zeta \rightarrow 0$ ,  $\varepsilon \rightarrow \bar{\varepsilon}$  if  $\sigma = 0$ , otherwise  $\varepsilon \rightarrow \infty$ . The Casimir (Lifshitz) free energy between two half spaces, separated by a distance  $a$ , assumed to be of the same material for simplicity, is given by

$$F = \frac{T}{2\pi} \sum'_{m=0} \int_{\zeta_m}^{\infty} d\kappa \kappa [\ln(1 - r_{\text{TM}}^2 e^{-2\kappa a}) + \ln(1 - r_{\text{TE}}^2 e^{-2\kappa a})]. \quad (1.2)$$

Here  $\kappa$ ,  $r_{\text{TE}}$ , and  $r_{\text{TM}}$  are functions of the discrete Matsubara frequencies  $\zeta_m = 2\pi mT$ ;  $\kappa^2 = k_{\perp}^2 + \zeta^2$  with  $\mathbf{k}_{\perp}$  the transverse wave vector, directed parallel to the surfaces. As is conventional, the prime on the summation mark implies the  $m=0$  term be taken with half weight. We need to examine the behavior of the reflection coefficients in the small  $\zeta$  limit. These are

$$r_{\text{TE}} = \frac{\kappa - \sqrt{\kappa^2 + \zeta^2}(\varepsilon - 1)}{\kappa + \sqrt{\kappa^2 + \zeta^2}(\varepsilon - 1)}, \quad (1.3)$$

$$r_{\text{TM}} = \frac{\varepsilon\kappa - \sqrt{\kappa^2 + \zeta^2}(\varepsilon - 1)}{\varepsilon\kappa + \sqrt{\kappa^2 + \zeta^2}(\varepsilon - 1)}, \quad (1.4)$$

where  $\varepsilon = \varepsilon(i\zeta)$ . For the case of an ideal metal, it was  $r_{\text{TE}}$  which was discontinuous:

$$r_{\text{TE}}(\zeta = 0) = 0, \quad \lim_{\zeta \rightarrow 0} r_{\text{TE}} = -1, \quad (1.5)$$

so this gave a linear temperature term when the sum over Matsubara frequencies is converted to an integral according to the Euler-Maclaurin formula, for example (Ref. [19] uses the Abel-Plana formula, but that is equivalent).

For a dielectric the TE reflection coefficient is continuous and vanishes as  $\zeta \rightarrow 0$ , but if there is a small (but not zero) conductivity which vanishes with  $T$  linearly or faster, the TM coefficient exhibits a discontinuity at  $\zeta = 0$  as we now explain. When the conductivity is small we can assume there exists a temperature so that the  $m=1$  Matsubara frequency,  $\zeta_1 = 2\pi T$ , satisfies the inequality

$$0 < 4\pi\sigma \ll \zeta_1 \ll \omega_0, \quad (1.6)$$

in which case

$$r_{\text{TM}}(i\zeta = 0) = 1, \quad r_{\text{TM}}(i\zeta_1) = \frac{\bar{\varepsilon} - 1}{\bar{\varepsilon} + 1}. \quad (1.7)$$

Typical values of  $\omega_0$  are in the optical or near IR frequency regions, so Eq. (1.6) will hold at room temperature for many semiconductors. If now  $\sigma$  goes to zero as  $T \rightarrow 0$  linearly or faster, Eq. (1.6) continues to hold true all the way to zero temperature where Eq. (1.7) becomes a true discontinuity,

$$r_{\text{TM}}(i\zeta = 0) = 1, \quad \lim_{\zeta \rightarrow 0} r_{\text{TM}}(i\zeta) = \frac{\bar{\varepsilon} - 1}{\bar{\varepsilon} + 1}. \quad (1.8)$$

Clearly if  $\sigma$  reaches some residual value  $> 0$ , Eq. (1.6) will not hold near zero temperature. Likewise the discontinuity disappears should  $\sigma$  be exactly zero in a temperature region of finite width including  $T = 0$ .

As in the metal case, Eq. (1.8) gives rise to a linear temperature term in the pressure and the free energy (see, e.g., Ref. [17] and references therein for details). Let  $f_m$  be the summand of Eq. (1.2) or a similar expression for the Casimir pressure. Since  $f_m$  is discontinuous at  $m=0$ , we must replace it by a continuous function,

$$\sum'_{m=0} f_m = \frac{1}{2}f_0 + \sum_{m=1}^{\infty} f_m = \frac{1}{2}(f_0 - \tilde{f}_0) + \sum'_{m=0} \tilde{f}_m, \quad (1.9)$$

where  $\tilde{f}_m$  is continuous,

$$\tilde{f}_m = \begin{cases} \tilde{f}_0 & = \lim_{m \rightarrow 0} f_m, & m = 0, \\ \tilde{f}_m & = f_m, & m \neq 0, \end{cases} \quad (1.10)$$

so that the Euler-Maclaurin summation formula can be applied to the sum over  $\tilde{f}_m$ . Then the first term in the third form in Eq. (1.9) gives rise to a free-energy contribution which is a linear function of  $T$ . Defining the shorthand notation

$$A_0 = \left( \frac{\bar{\varepsilon} - 1}{\bar{\varepsilon} + 1} \right)^2, \quad (1.11)$$

that linear term is

$$\begin{aligned} F^{\text{TM}} &= \frac{T}{4\pi} \int_0^{\infty} d\kappa \kappa [\ln(1 - A_0 e^{-2\kappa a}) - \ln(1 - e^{-2\kappa a})] \\ &= \frac{T}{4\pi} \sum_{n=1}^{\infty} \frac{1}{n} [A_0^n - 1] \int_0^{\infty} d\kappa \kappa e^{-2n\kappa a} \\ &= \frac{T}{16\pi a^2} [\text{Li}_3(A_0) - \zeta(3)], \end{aligned} \quad (1.12)$$

where the polylogarithmic function is

$$\text{Li}_n(\xi) = \sum_{k=1}^{\infty} \frac{\xi^k}{k^n}. \quad (1.13)$$

Note that the linear term vanishes for  $\bar{\varepsilon} \rightarrow \infty$  as is clear from noting the relation to the Riemann  $\zeta$  function,

$$\text{Li}_n(1) = \zeta(n). \quad (1.14)$$

Thus at zero temperature, the entropy is nonzero,

$$S = - \left( \frac{\partial F}{\partial T} \right)_V = - \frac{1}{16\pi a^2} [\text{Li}_3(A_0) - \zeta(3)], \quad (1.15)$$

which, if physical, is a violation of the Nernst heat theorem, or the third law of thermodynamics, which states that the entropy of a system must vanish at zero temperature.

## II. GENERAL FORMALISM

For reference throughout the next sections we will go through the formalism of determining the leading temperature corrections to the Casimir (Lifshitz) free energy by use of the Euler-Maclaurin formula, a procedure often employed previously.

Considering one polarization mode at a time, the free energy for the  $q$  mode ( $q$ =TM, TE) is written in the form

$$F_q = f(a, T) \sum_{m=0}^{\infty} g(m), \quad (2.1)$$

where we have pulled out a convenient prefactor.

When  $T \rightarrow 0$  the Matsubara sum becomes an integral, so the temperature correction to the free energy, given by

$$\Delta F_q = f(a, T) \left[ \sum_{m=0}^{\infty} g(m) - \int_0^{\infty} dm g(m) \right], \quad (2.2)$$

can be determined by use of the Euler-Maclaurin formula. For the summands of the Lifshitz formula, the higher derivatives of  $g(m)$  are singular near  $m=0$ . When this is the case the Euler-Maclaurin formula can be applied to the sum starting at  $m=1$  (or a higher value of  $m$ ) instead, whereby

$$\begin{aligned} \tilde{\Gamma} &\equiv \left[ \sum_{m=0}^{\infty} g(m) - \int_0^{\infty} dm g(m) \right] \\ &= \frac{1}{2}g(0) - \int_0^1 g(m)dm + \frac{1}{2}g(1) - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} g^{(2k-1)}(1) \\ &= \frac{1}{2}g(0) - \int_0^1 g(m)dm + \frac{1}{2}g(1) - \frac{1}{12}g'(1) + \frac{1}{720}g'''(1) \\ &\quad - \dots, \end{aligned} \quad (2.3)$$

where  $B_n$  are the Bernoulli numbers,

$$B_2 = \frac{1}{6}; \quad B_4 = -\frac{1}{30}; \quad B_6 = \frac{1}{42}; \quad \dots, \quad (2.4)$$

using the convention of [24] Sec. 23.2. (Two remarks are called for here: We have assumed that  $g$  and all its derivatives vanish at infinity, and we have converted this formula into one which is commonly asymptotic because we have omitted the remainder term which is present when only a finite number of derivatives terms are retained. Thus we are considering only the leading terms in an asymptotic expansion for small  $T$ .)

As mentioned above,  $g(m)$  is not analytic at  $m=0$ . It can be written in the asymptotic form for small  $m$

$$g(m) \sim c_0 + c_1 m + c_{3/2} m^{3/2} + c_{2l} m^2 \ln m + c_2 m^2 + \dots, \quad (2.5)$$

$$m \rightarrow 0.$$

The terms needed for the right-hand side of Eq. (2.3) are now

$$g(0) = c_0, \quad (2.6a)$$

$$g(1) = c_0 + c_1 + c_2 + c_{3/2} + \dots, \quad (2.6b)$$

$$g'(1) = c_1 + c_{2l} + 2c_2 + \frac{3}{2}c_{3/2} + \dots, \quad (2.6c)$$

$$g'''(1) = 2c_{2l} - \frac{3}{8}c_{3/2} + \dots, \quad (2.6d)$$

$$\int_0^1 dm g(m) = c_0 + \frac{1}{2}c_1 - \frac{1}{9}c_{2l} + \frac{1}{3}c_2 + \frac{2}{5}c_{3/2} + \dots. \quad (2.6e)$$

When inserted into Eq. (2.3) the terms involving  $c_0$  and  $c_2$  cancel and one is left with

$$\tilde{\Gamma} \approx -\frac{c_1}{12} + \frac{11c_{2l}}{360} - \frac{49}{1920}c_{3/2} + \dots. \quad (2.7)$$

Here the term due to  $c_1$  is exact, whereas the terms with  $c_{2l}$  and  $c_{3/2}$  receive contributions from all higher derivatives in the Euler-Maclaurin formula, and to obtain exact expressions for the coefficients, all such terms must be kept, as we now show.

Retaining the higher derivative terms in the Euler-Maclaurin formula one finds by using

$$\phi_{2n} = \frac{d^{2n-1}}{dm^{2n-1}} m^{3/2} \Big|_{m=1} = -\frac{3(4n-7)!}{2^{4n-5}(2n-4)!}, \quad n \geq 2, \quad (2.8a)$$

and

$$\psi_{2n} = \frac{d^{2n-1}}{dm^{2n-1}} m^2 \ln m \Big|_{m=1} = 2(2n-4)!, \quad n \geq 2, \quad (2.8b)$$

that the temperature correction to free energy is

$$\Delta F_q = f(a, T) \tilde{\Gamma}, \quad (2.9)$$

where with Eq. (2.3),

$$\tilde{\Gamma} = -\frac{c_1}{12} + \Psi c_{2l} + \Phi c_{3/2} + \dots, \quad (2.10)$$

with the coefficients

$$\Psi = \frac{1}{9} - \frac{B_2}{2} - \sum_{n=2}^{\infty} \frac{B_{2n} \psi_{2n}}{(2n)!}, \quad (2.11a)$$

$$\Phi = \frac{1}{2} - \frac{2}{5} - \frac{3B_2}{4} - \sum_{n=2}^{\infty} \frac{B_{2n}\phi_{2n}}{(2n)!}. \quad (2.11b)$$

These series are formally divergent as is typical for perturbation series near singularities. Indeed, they arise from the asymptotic Euler-Maclaurin formula (2.3). For example,  $\Phi$  can be recognized as a special case of the expansion of the Riemann  $\zeta$  function in terms of Bernoulli numbers, Eq. 23.2.3 of Ref. [24] with an infinite number of terms retained in the sum, and the remainder omitted. A meaningful value can nonetheless be assigned to them through Borel summation as detailed in the Appendix. Numerically, the mathematical software Maple computes the numerical values by means of a Levin  $u$  transform to

$$\Psi = 0.03044845705840 \dots, \quad (2.12a)$$

$$\Phi = -0.0254852018898 \dots. \quad (2.12b)$$

By either numerical or analytical correspondence we thus recognize that

$$\Phi = \zeta\left(-\frac{3}{2}\right), \quad (2.13a)$$

$$\Psi = \frac{\zeta(3)}{4\pi^2}, \quad (2.13b)$$

where  $\zeta$  is the Riemann  $\zeta$  function.

When Eq. (1.1) is used in the Lifshitz formalism with constant and finite  $\sigma$  and  $\bar{\epsilon} > 1$  in Secs. III and IV, we will find that the terms of  $F$  stemming from  $c_1$ ,  $c_{3/2}$ , and  $c_{2l}$  are proportional to  $T^2$ ,  $T^{5/2}$ , and  $T^3$ , respectively. Higher-order terms of  $g(m)$  will likewise give higher-order temperature corrections.

### III. TM MODE, RESIDUAL CONDUCTIVITY

In the following sections we will work out the low-temperature behavior of corrections to the free energy under the assumption that a Drude-type dielectric function (1.1) may be used, and that  $\sigma$  is finite and constant with respect to  $\zeta$  and  $T$  for small  $T$  and  $\zeta$ . As argued in Ref. [25], when  $\sigma$  is finite close to zero temperature Nernst's theorem will be satisfied. Here we will calculate explicitly the low-temperature behavior of the free energy for the TM mode.

Conventionally, semiconductors are found within the broad interval of conductivity  $\sigma^{\text{SI}}$  in SI units  $10^{-5} (\Omega \text{ m})^{-1} < \sigma^{\text{SI}} < 10^5 (\Omega \text{ m})^{-1}$ , that is

$$10^6 \text{ s}^{-1} < \sigma^{\text{SI}}/\epsilon_0 < 10^{16} \text{ s}^{-1}. \quad (3.1)$$

For numerical purposes we will use the intermediate value  $\sigma^{\text{SI}}/\epsilon_0 = 10^{12} \text{ s}^{-1}$ , which is large enough not to hamper numerical verification unnecessarily, but small enough to distinguish the material in question from a good metal. The frequency corresponding to  $\sigma^{\text{SI}}/\epsilon_0$  for a metal is  $\omega_p^2/\nu$ , where  $\omega_p$  is the plasma frequency and  $\nu$  the relaxation frequency. For gold at room temperature  $\omega_p^2/\nu$  has the approximate value  $3.5 \times 10^{22} \text{ s}^{-1}$ .

Returning to Gaussian units, we consider the TM mode and introduce the shorthand notation

$$t = \frac{\zeta_1}{4\pi\sigma} = \frac{2\pi T}{4\pi\sigma} = \frac{2\pi k_B T}{\hbar(\sigma^{\text{SI}}/\epsilon_0)} \quad (3.2)$$

and the symbol

$$\mu = mt. \quad (3.3)$$

If  $4\pi\sigma = 10^{12} \text{ s}^{-1}$  as assumed above,

$$t \approx 0.83T, \quad (3.4)$$

with  $T$  in Kelvin.

The free energy is given by Eq. (1.2), for which we now consider only the TM term,

$$F^{\text{TM}} = \frac{T}{2\pi} \sum_{m=0}^{\infty} \int_{\zeta}^{\infty} d\kappa \kappa \ln(1 - Ae^{-2\kappa a}), \quad (3.5)$$

where the reflection coefficient squared is

$$A \equiv r_{\text{TM}}^2 = \left( \frac{\epsilon - \sqrt{1 + (\epsilon - 1)(\zeta/\kappa)^2}}{\epsilon + \sqrt{1 + (\epsilon - 1)(\zeta/\kappa)^2}} \right)^2. \quad (3.6)$$

Here and henceforth the index  $m$  on Matsubara frequencies  $\zeta_m$  and quantities dependent on it will frequently be suppressed.

The temperature corrections to the free energy at low temperatures are dominated by small frequencies, so we can assume as an approximation that the middle term of Eq. (1.1) is simply equal to  $\bar{\epsilon} - 1$  and write

$$\epsilon(i\zeta) \approx \bar{\epsilon} + \frac{4\pi\sigma}{\zeta} = \bar{\epsilon} + \frac{1}{\mu}. \quad (3.7)$$

We define the dimensionless quantity

$$\alpha = 2a(4\pi\sigma) = \frac{2a}{c}(\sigma^{\text{SI}}/\epsilon_0), \quad (3.8)$$

where  $a$  is the distance between the semiconductor plates. For the value  $4\pi\sigma \approx 10^{12} \text{ s}^{-1}$  or smaller,  $\alpha$  is a small quantity, which we use to define a criterion for the smallness of the conductivity in the remainder of this paper,

$$\alpha \ll 1. \quad (3.9)$$

For  $a = 1 \mu\text{m}$  and  $\sigma$  as above, as used for numerical purposes later,  $\alpha$  has a value of about  $6.7 \times 10^{-3}$ , so this criterion is well satisfied.

By defining the variable  $x$ ,

$$x = 2\kappa a = \frac{\kappa\alpha}{4\pi\sigma} = \frac{\kappa\alpha\mu}{\zeta}, \quad (3.10)$$

$A$  can be written

$$A = \left( \frac{1 + \bar{\epsilon}\mu - \mu\sqrt{1 + [1 + (\bar{\epsilon} - 1)\mu]\alpha^2\mu/x^2}}{1 + \bar{\epsilon}\mu + \mu\sqrt{1 + [1 + (\bar{\epsilon} - 1)\mu]\alpha^2\mu/x^2}} \right)^2, \quad (3.11)$$

and the integral (3.5) with the use of Eq. (3.2) and  $\zeta = 2\pi mT$  becomes

$$F^{\text{TM}} \equiv \frac{(4\pi\sigma)^3 t}{4\pi^2 \alpha^2} \sum_{m=0}^{\infty} g(m), \quad (3.12)$$

where

$$g(m) = \int_{\alpha\mu}^{\infty} dx x \ln(1 - Ae^{-x}). \quad (3.13)$$

We wish now to extract explicitly the temperature dependence of the integrals  $g(m)$  in Eq. (3.13). The procedure we choose is to expand Eq. (3.13) to leading order in the small parameter  $\alpha$ , and then expand the resulting term in powers of  $m$  to obtain the form (2.5).

The first term in the Taylor expansion of the logarithm in powers of  $\alpha$  is

$$\ln(1 - Ae^{-x}) = -\text{Li}_1(A_\mu e^{-x}) + O(\alpha^2) \quad (3.14)$$

where we use the polylogarithmic function defined in Eq. (1.13) and define the quantity

$$A_\mu = \left( \frac{1 + (\bar{\varepsilon} - 1)\mu}{1 + (\bar{\varepsilon} + 1)\mu} \right)^2. \quad (3.15)$$

For integral  $s \leq 1$  the polylogarithm  $\text{Li}_s(y)$  can be expressed by elementary functions, specifically

$$\text{Li}_1(y) = -\ln(1 - y); \quad \text{Li}_0(y) = \frac{y}{1 - y};$$

$$\text{Li}_{-1} = \frac{y}{(1 - y)^2}. \quad (3.16)$$

The summand  $g(m)$  thus has the form

$$g(m) = - \int_{\alpha\mu}^{\infty} dx x \text{Li}_1(A_\mu e^{-x}) + O(\alpha^2). \quad (3.17)$$

Now we will expand  $g(m)$  in powers of  $m$ . It is easy to show from Eq. (1.13) that

$$\int dy \text{Li}_n(Ce^{-\beta y}) = -\frac{1}{\beta} \text{Li}_{n+1}(Ce^{-\beta y}); \quad (3.18)$$

from which by partial integration

$$g(m) = -\alpha\mu \text{Li}_2(A_\mu e^{-\alpha\mu}) - \text{Li}_3(A_\mu e^{-\alpha\mu}) + O(\alpha^2). \quad (3.19)$$

We now use the property

$$\text{Li}_n(Ce^{-y}) = \sum_{l=0}^{\infty} \frac{(-y)^l}{l!} \text{Li}_{n-l}(C) \quad (3.20)$$

for  $|C| < 1$  to expand the polylogarithms in powers of  $\alpha\mu$ . The terms containing  $\text{Li}_2$  then cancel and we are left with

$$g(m) = -\text{Li}_3(A_\mu) + \frac{1}{2} \alpha^2 \mu^2 \text{Li}_1(A_\mu) + O(\alpha^2) \quad (3.21)$$

with  $A_\mu$  given by Eq. (3.15). Henceforth we shall denote the first two terms of the expansion (3.21)  $g_I(m)$  and  $g_{II}(m)$ . The remaining  $O(\alpha^2)$  term comes from the error in Eq. (3.14). As

before we are going to truncate the expansion in  $\alpha$  at leading order, but will evaluate the explicit correction  $\sim \alpha^2$  to Eq. (3.21) later as a measure of the error. We thus have the simple expression

$$g_I(m) = -\text{Li}_3(A_\mu). \quad (3.22)$$

We will next expand Eq. (3.22) in  $\mu$ .  $\text{Li}_3(A_\mu)$  does not have a Taylor expansion near  $m=0$  (where  $A_0=1$ ) because its second derivative is singular here. Using

$$\frac{d}{dy} \text{Li}_n(y) = \frac{1}{y} \text{Li}_{n-1}(y), \quad (3.23)$$

we differentiate Eq. (3.22) to find

$$g'_I(m) = \frac{4t \text{Li}_2(A_\mu)}{[1 + (\bar{\varepsilon} + 1)\mu][1 + (\bar{\varepsilon} - 1)\mu]}. \quad (3.24)$$

We can use the identity [26]

$$\text{Li}_2(z) + \text{Li}_2(1 - z) = \frac{\pi^2}{6} - \ln(z)\ln(1 - z), \quad (3.25)$$

which is easily verified by differentiation, use of Eq. (1.13), and  $\text{Li}_2(1) = \pi^2/6$ . Furthermore,  $\text{Li}_2(1 - A_\mu)$  has a simple Taylor expansion around  $A_\mu = 1$ ,

$$\text{Li}_2(1 - A_\mu) = 4\mu - 4(\bar{\varepsilon} + 1)\mu^2 + \dots \quad (3.26)$$

and

$$\frac{4}{[1 + (\bar{\varepsilon} + 1)\mu][1 + (\bar{\varepsilon} - 1)\mu]} = 4 - 8\bar{\varepsilon}\mu + \dots, \quad (3.27)$$

whereby we find

$$g'_I(m) = \frac{2\pi^2 t}{3} - 4mt^2 \left( \frac{\bar{\varepsilon}\pi^2}{3} + 4 \right) + 16mt^2 \ln 4\mu + \dots, \quad (3.28)$$

where the next term of the series is of order  $t^3$ .

Comparing with Eq. (2.5) we recognize the coefficients

$$c_1 = \frac{2\pi^2 t}{3}, \quad c_{2l} = 8t^2, \quad (3.29)$$

which we insert into Eq. (2.10) to find

$$\left[ \sum_{m=0}^{\infty} - \int_0^{\infty} dm \right] g_I(m) = -\frac{\pi^2 t}{18} + 8\Psi t^2. \quad (3.30)$$

We thus obtain the approximate correction to the free energy for small  $t$ ,

$$\begin{aligned} \Delta F_I^{\text{TM}} &= \frac{(4\pi\sigma)^3 t}{4\pi^2 \alpha^2} \left[ \sum_{m=0}^{\infty} - \int_0^{\infty} dm \right] g_I(m) \\ &\approx -\frac{(4\pi\sigma)^3}{72\pi^2 \alpha^2} t^2 [\pi^2 - 144\Psi t] \end{aligned} \quad (3.31)$$

in terms of our reduced units  $t$  and  $\alpha$ . In SI units, inserting Eq. (2.13b),



$$\begin{aligned}\Delta F_I^{\text{TM}} &= -\frac{\pi^2(k_B T)^2}{72\hbar(\sigma^{\text{SI}}/\epsilon_0)a^2} + \frac{\zeta(3)(k_B T)^3}{\pi[\hbar(\sigma^{\text{SI}}/\epsilon_0)a]^2} \\ &= -\frac{\pi^2(k_B T)^2}{72\hbar(\sigma^{\text{SI}}/\epsilon_0)a^2} \left(1 - \frac{72\zeta(3)k_B T}{\pi^3\hbar\sigma^{\text{SI}}/\epsilon_0}\right).\end{aligned}\quad (3.32)$$

### A. Correction due to subleading terms of Eq. (3.21)

Twice in the above we truncated the expressions at leading order in the parameter  $\alpha$ , in Eq. (3.14) and Eq. (3.21). As an indication of the magnitude of the error we will calculate the next order in  $\alpha$  of Eq. (3.21) while a similar calculation for Eq. (3.14) is more troublesome due to singularities and beyond the scope of the present effort. The correction  $\propto \alpha^2$  of Eq. (3.21) was

$$\Delta g(m) = \frac{1}{2}\alpha^2 m^2 t^2 \text{Li}_1(A_\mu) + O(\alpha^3 t^3). \quad (3.33)$$

We will only consider the first term, since the next terms give temperature corrections  $\propto T^4$  and higher. We Taylor expand as before in powers of  $\mu$ ,

$$\text{Li}_1(A_\mu) = -\ln(4t) - \ln m + (\bar{\epsilon} + 2)\mu + O(\mu^2), \quad (3.34)$$

from which the leading correction from  $\Delta g_I(m)$ , is found from Eq. (2.10) to order  $T^3$  to which only the term  $\propto m^2 \ln m$  contributes,

$$\delta F^{\text{TM}} = \frac{(4\pi\sigma)^3 t}{4\pi^2 \alpha^2} \delta \tilde{\Gamma} \approx \frac{(4\pi\sigma)^3 \Psi t^3}{8\pi^2}. \quad (3.35)$$

Being  $\alpha$  independent, the correction (3.35) is much smaller than the leading term (3.31) for small  $\alpha$ . In SI units,

$$\delta F^{\text{TM}} = \frac{\zeta(3)(k_B T)^3}{4\pi\hbar^2 c^2} + O(T^4). \quad (3.36)$$

The relative magnitude of this term compared to the  $T^3$  term of Eq. (3.32) is with our numerical data

$$\frac{(\sigma^{\text{SI}}/\epsilon_0)^2 a^2}{4c^2} \approx 2.8 \times 10^{-6}. \quad (3.37)$$

The correction from the truncation of Eq. (3.14) is likely to be of similar size and therefore much smaller than the accuracy of the numerical investigation.

### B. Numerical investigation of TM mode result

The numerical investigation in Fig. 1(a) employs Eq. (1.1) with  $\sigma^{\text{SI}}/\epsilon_0 = 10^{12} \text{ s}^{-1}$ ,  $\bar{\epsilon} = 11.67$ , and  $\omega_0 = 8 \times 10^{15} \text{ s}^{-1}$  as appropriate for Si [27]. While the analytical expression fits well for  $T < 0.1$  K, corrections  $\sim T^4$  become important beyond this point. The two leading orders in temperature corrections were shown to be independent of  $\bar{\epsilon}$  to leading order in  $\alpha$ , but the  $T^4$  correction (not calculated analytically herein) depends heavily on this value. A qualitative measure of this effect is given in Fig. 1(b) where we have used  $\bar{\epsilon} = 1$ , cetera paribus. In all plots the curves are given by Eq. (3.32) with and without inclusion of its second term.

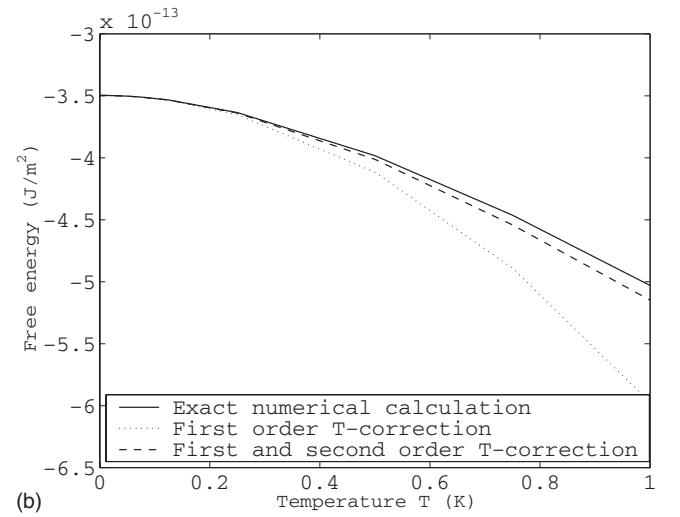
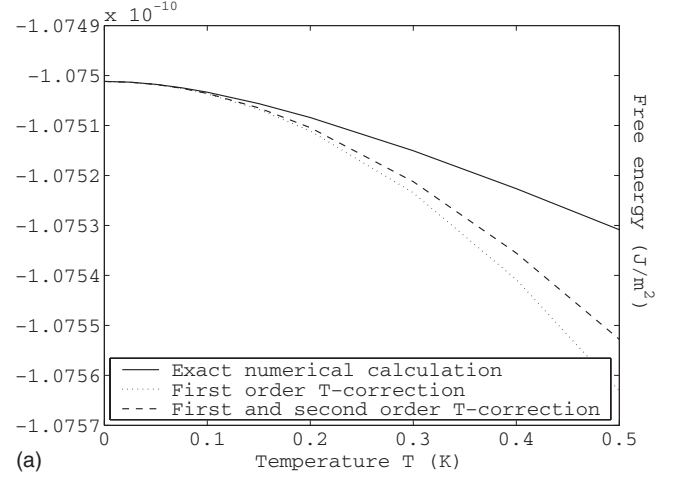


FIG. 1.  $F^{\text{TM}}$  and its approximation with (a)  $\bar{\epsilon} = 11.67$  and (b)  $\bar{\epsilon} = 1$ . Correction curves, calculated from Eq. (3.31), are shifted to match the numerical calculations at  $T=0$  in each graph.

It is noteworthy that, as seen from Fig. 1, while the  $\zeta^{-1}$  term of Eq. (1.1) gives the dominant temperature correction for small  $T$ , nearly all (99.7% with our data) of the free energy at  $T=0$  is due to the  $\bar{\epsilon}$  term.

While the fit pictured in Fig. 1 is indicative, a much more sensitive confirmation of the accuracy of the theoretical results is provided by considering the quantity

$$R = \frac{\Delta F_{\text{th}} - \Delta F_{\text{num}}}{\Delta F_{\text{th}}}, \quad (3.38)$$

where  $\Delta F_{\text{num}}$  is the direct numerical calculation and  $\Delta F_{\text{th}}$  is the theoretical result to next-to-leading order, in the form (3.31). An analysis exactly similar to this was performed in Ref. [23]; the reader may refer to that paper for further details.

We have found that  $\Delta F_{\text{th}}$  is of the form

$$\Delta F_{\text{th}} = -CT^2(1 - C_1 T) \quad (3.39)$$

and assume  $\Delta F_{\text{num}}$  to be of the form

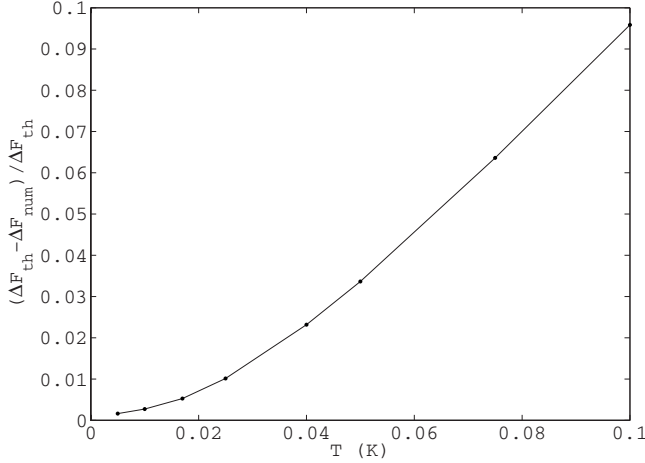


FIG. 2. The quantity  $R$  defined in Eq. (3.38) plotted for the TM result (3.31) and numerical calculations.

$$\Delta F_{\text{num}} = -DT^2(1 - D_1T + D_2T^2 + \dots), \quad (3.40)$$

from which one finds

$$R = \frac{C-D}{C} - \frac{D}{C}(C_1 - D_1)T - \frac{D}{C}[D_2 + C_1(C_1 - D_1)]T^2 + \dots \quad (3.41)$$

In the special case where  $C=D$  and  $C_1=D_1$ , this becomes

$$R = -D_2T^2 + O(T^3) \quad (3.42)$$

which is zero in the limit  $T=0$  and has zero slope in this limit. The zero-temperature limit of  $R$  and its slope thus provide measures of the accuracy of the theoretical results: if the  $T^2$  coefficient is correct,  $R$  should approach zero as  $T \rightarrow 0$ , and if the  $T^3$  coefficient is correct, the slope of  $R(T)$  should vanish in this limit as well. We have not taken the corrections (3.36) into account in the plotting of Fig. 2.

We have undertaken a numerical study of the behavior close to zero temperature, resulting in the graph of  $R$  shown in Fig. 2. Due to the vanishing denominator of Eq. (3.38), the analysis is extremely sensitive to numerical errors as the zero temperature limit is approached. From the figure it seems clear that the errors in the two coefficients are small enough to confirm the correctness of Eq. (3.31), although some caution must be exerted due to the numerical volatility of  $R$ . Comparing Fig. 2 to Eq. (3.42) it is clear that  $D_2 < 0$  which implies that the coefficient of the  $T^4$  term of the free energy be positive, which conforms with the corrections in Fig. 1 not accounted for to order  $T^3$ .

#### IV. TE MODE, RESIDUAL CONDUCTIVITY

For the TE mode the dominant temperature correction to the free energy comes from the last term of Eq. (1.1). The permittivity (1.1), which can be approximated as Eq. (3.7), is similar, but not identical, to that for a Drude metal, considered in Refs. [23,28]. There, instead of Eq. (3.7) the permittivity was assumed to be

$$\epsilon_{\text{metal}} = 1 + \frac{\omega_p^2}{\zeta(\zeta + \nu)} \approx 1 + \frac{\omega_p^2}{\nu\zeta}. \quad (4.1)$$

The principal difference is that the constant term  $\bar{\epsilon}$  is assumed to be significant here and kept general. Since for small  $\zeta$  the term  $\sim \zeta^{-1}$  dominates the constant term, an approximation to the low-temperature behavior of the dielectric would be expected to be found by the same analysis as that of Refs. [23,28] but with the substitution

$$\frac{\omega_p^2}{\nu} \rightarrow 4\pi\sigma. \quad (4.2)$$

For typical semiconductors,  $4\pi\sigma$  is smaller than  $\omega_p^2/\nu$  for a good metal by many orders of magnitude. For this reason, since the free energy at zero temperature is of the same order of magnitude for the metals and semiconductors for the same separation, the relative temperature corrections for the TE mode are expected to be much smaller than for a metal. Thus, there is reason to investigate whether the effects of  $\bar{\epsilon} > 1$ , while negligible for a metal, could be important for small  $\sigma$ . In some dielectric materials, as is well known,  $\bar{\epsilon}$  can exceed unity by as much as two orders of magnitude, and a more careful analysis is therefore justified. The procedure is the same as above, and an extension of that found in Ref. [23], to which the reader may turn for further detail.

It was found in Refs. [23,28] that for  $T \rightarrow 0$ , and  $\bar{\epsilon} = 1$ ,

$$\Delta F_{\text{TE}} = C_2 T^2 - C_{5/2} T^{5/2} + \dots, \quad (4.3)$$

where

$$C_2 = \frac{(4\pi\sigma)}{48} (2 \ln 2 - 1), \quad (4.4a)$$

$$C_{5/2} = \frac{\sqrt{2\pi}}{6} \zeta(-3/2) (4\pi\sigma)^{3/2} a. \quad (4.4b)$$

Here  $\zeta(y)$  is the Riemann  $\zeta$  function (for this closed form of  $C_{5/2}$ , see Appendix A of Ref. [23]).

For the numerical values indicated this gives the SI values

$$C_2 = 1.618\,571\,9 \times 10^{-19} \frac{\text{J}}{\text{K m}^2} \left( \frac{\sigma^{\text{SI}}/\epsilon_0}{10^{12} \text{ s}^{-1}} \right), \quad (4.5a)$$

$$C_{5/2} = 2.584\,437\,3 \times 10^{-22} \frac{\text{J}}{\text{K}^{5/2} \text{ m}^2} \left( \frac{a}{1 \mu\text{m}} \right) \left( \frac{\sigma^{\text{SI}}/\epsilon_0}{10^{12} \text{ s}^{-1}} \right)^{3/2}. \quad (4.5b)$$

Thus the TE temperature correction is expected to be positive and in the order of magnitude of  $10^{-19} \text{ J/m}^2$  at  $T=1 \text{ K}$ .

The numerical calculations shown in Fig. 3 were complicated by the fact that the thermal corrections are many orders of magnitude smaller than the free energy at zero temperature, making a graph of the quantity  $R$  similar to Fig. 2 unfeasible within the assumption of  $\alpha \ll 1$ . We show here, however, that assuming  $\bar{\epsilon} > 1$  does not change the theoretically predicted thermal correction to the free energy to order  $T^3$ , and therefore merely refer to Ref. [23] for further numerical support of the theoretical result.

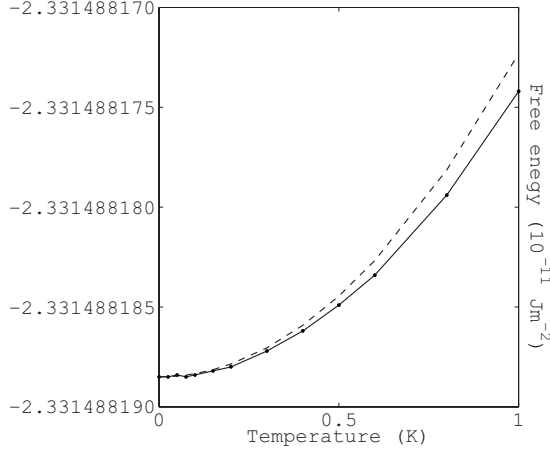


FIG. 3. Temperature dependence of the free energy for TE at 1 micron separation. The solid line is an exact numerical calculation including all terms of Eq. (1.1) with  $\bar{\varepsilon}=11.66$  and  $\omega_0=8.0 \times 10^{15} \text{ s}^{-1}$ , the dashed line is the parabolic temperature correction (4.3). The term  $\propto T^{5/2}$  is too small to be visible in the graph.

#### A. More general treatment of the TE mode

Let us treat the TE mode temperature correction to the free energy more carefully. Starting with the expression (1.2) we perform the substitution

$$x = \frac{\kappa}{\zeta \sqrt{\varepsilon(i\zeta)} - 1} = \frac{\kappa\mu}{\chi\zeta} \quad (4.6)$$

where we define the recurring quantity

$$\chi = \sqrt{\mu + (\bar{\varepsilon} - 1)\mu^2}. \quad (4.7)$$

Then the free energy may be written

$$F^{\text{TE}} = \frac{(4\pi\sigma)^3 t}{4\pi^2} \sum_{m=0}^{\infty} g(m) \quad (4.8a)$$

with

$$g(m) = \chi^2 \int_{\mu/\chi}^{\infty} dx x \ln(1 - B e^{-\alpha x}). \quad (4.8b)$$

The squared reflection coefficient given by Eq. (1.3) now depends only on  $x$ ,

$$B = (x - \sqrt{x^2 + 1})^4. \quad (4.9)$$

We expand the integrand of  $g(m)$ ,

$$\ln(1 - B e^{-\alpha x}) = \ln(1 - B) + \frac{\alpha \chi x B}{1 - B} + \dots \quad (4.10)$$

Note that this is as far as we can expand this way, since the next term of the  $\alpha$  expansion gives a divergent contribution (an alternative method which avoids some divergences but is somewhat more cumbersome is the method employed in Appendix A of Ref. [23] where the corrections are calculated without the use of the Euler-Maclaurin formula).

Consider the first terms of the expansion (4.10) [we dub the terms of  $g(m)$  from the expansion  $g_I(m), g_{II}(m), \dots$ ],

$$g_I(m) = \chi^2 \int_{\mu/\chi}^{\infty} dx x \ln[1 - (x - \sqrt{x^2 + 1})^4]. \quad (4.11)$$

This integral can be evaluated explicitly (a similar integral was evaluated in Ref. [23] where the lower limit was approximated as zero). Perform the substitution  $x = \sinh u$ . Then we may write

$$g_I(m) = \frac{\chi^2}{4} \int_{u_0}^{\infty} du (e^{2u} - e^{-2u}) \ln(1 - e^{-4u}) \quad (4.12a)$$

with

$$u_0 = \text{arsinh} \frac{\mu}{\chi} = \frac{1}{2} \ln \left( \frac{\sqrt{\bar{\varepsilon}\mu + 1} + \sqrt{\mu}}{\sqrt{\bar{\varepsilon}\mu + 1} - \sqrt{\mu}} \right). \quad (4.12b)$$

With the substitution  $y = e^{-2u}$ ,

$$g_I(m) = \frac{\chi^2}{8} \int_0^{y_0} dy (y^{-2} - 1) \ln(1 - y^2), \quad (4.13a)$$

where

$$\begin{aligned} y_0 = e^{-2u_0} &= \frac{\sqrt{\bar{\varepsilon}\mu + 1} - \sqrt{\mu}}{\sqrt{\bar{\varepsilon}\mu + 1} + \sqrt{\mu}} \\ &= 1 - 2\sqrt{\mu} + 2\mu + (\bar{\varepsilon} - 2)\mu^{3/2} + \dots \end{aligned} \quad (4.13b)$$

The integral is straightforward to evaluate and the result is

$$g_I(m) = -\frac{\chi^2}{8} \left[ \left( \frac{1}{y_0} + y_0 \right) \ln(1 - y_0^2) - 2y_0 + 2 \ln \frac{1 + y_0}{1 - y_0} \right]. \quad (4.14)$$

We expand this in powers of  $\mu$  and find that the terms  $\propto \mu^{3/2}$  cancel, consistent with the small- $x$  dependence of the integrand of  $g_I(m)$ . We are left with

$$\begin{aligned} g_I(m) &= -\frac{\mu}{4} (2 \ln 2 - 1) - \frac{\mu^2}{4} [\ln 4\mu + \bar{\varepsilon} (2 \ln 2 - 1)] \\ &\quad + \frac{2}{3} \mu^{5/2} + O(\mu^3). \end{aligned} \quad (4.15)$$

Comparing with Eq. (2.5) we see

$$c_1 = -\frac{t}{4} (2 \ln 2 - 1) \text{ and } c_{2I} = -\frac{t^2}{4}, \quad (4.16)$$

while the dependence on  $\bar{\varepsilon}$  only enters in the  $c_2$  term  $\sim m^2$  which does not contribute to the Euler-Maclaurin formula. The temperature correction to first order in  $\alpha$  is thus

$$\Delta F_I^{\text{TE}} \approx \frac{(4\pi\sigma)^3 t^2}{4\pi^2} \left[ \frac{2 \ln 2 - 1}{48} - \frac{\Psi t}{4} \right]. \quad (4.17)$$

We see that the leading term conforms with Eq. (4.3) when Eq. (4.4a) is inserted. The first term beyond those calculated is proportional to  $T^{7/2}$  according to Eq. (4.15). In SI units with (2.13b),

$$\Delta F_I^{\text{TE}} \approx \frac{\sigma^{\text{SI}} (k_B T)^2}{48 \varepsilon_0 \hbar c^2} (2 \ln 2 - 1) - \frac{\zeta(3) (k_B T)^3}{8 \pi \hbar^2 c^2}. \quad (4.18)$$



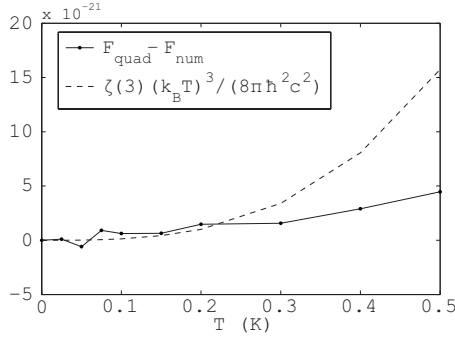


FIG. 4. The difference between the numerically calculated free energy and the quadratic  $T$  term of Eq. (4.18) (equal to the difference between the graphs in Fig. 3) plotted against the absolute value of the  $T^3$  term of Eq. (4.18).

The  $T^3$  term of Eq. (4.18) has the same form as that found for ideal metals in the limit  $aT \ll 1$  [29,30]. A similar term is present in Eq. (3.36). [Note that the  $T^3$  correction for the TM mode is not fully accounted for therein.] A numerical comparison of this term with the difference between the graphs in Fig. 3 is shown in Fig. 4. It shows that the  $T^3$  coefficient in Eq. (4.18) is the right order of magnitude, but the numerical precision is not sufficient to draw definite conclusions about its accuracy at this time.

#### B. First-order correction to expansion (4.10)

The first-order correction term in Eq. (4.10) is easily calculated with a similar scheme. We have

$$g_{II}(m) = \alpha \chi^3 \int_{\mu/\chi}^{\infty} dx \frac{x^2 B}{1-B} = \frac{\alpha \chi^3}{4} \int_{\mu/\chi}^{\infty} dx \frac{x(x - \sqrt{x^2 + 1})^2}{\sqrt{x^2 + 1}}. \quad (4.19)$$

The procedure for solving this integral is as before. Substitute  $x = \sinh u$  to obtain with a little shuffling

$$g_{II}(m) = \frac{\alpha \chi^3}{8} \int_{u_0}^{\infty} du e^{-u} (1 - e^{-2u}). \quad (4.20)$$

With the substitution  $z = e^{-u}$  this becomes very simple,

$$g_{II}(m) = -\frac{\alpha \chi^3}{8} \int_{z_0}^0 dz (1 - z^2) = \frac{\alpha \chi^3}{8} \left( z_0 - \frac{z_0^3}{3} \right) \quad (4.21)$$

with

$$\begin{aligned} z_0 = e^{-u_0} &= \left( \frac{\sqrt{\bar{\epsilon}\mu + 1} - \sqrt{\mu}}{\sqrt{\bar{\epsilon}\mu + 1} + \sqrt{\mu}} \right)^{1/2} \\ &= 1 - \sqrt{\mu} + \frac{\mu}{2} + \frac{1}{2}(\bar{\epsilon} - 1)\mu^{3/2} + \dots \end{aligned} \quad (4.22)$$

Thus we find the  $\mu$  expansion of  $g_{II}(m)$ ,

$$g_{II}(m) = \frac{\alpha}{8} \left( \frac{2}{3}\mu^{3/2} - (2 - \bar{\epsilon})\mu^{5/2} + \dots \right). \quad (4.23)$$

Hence, with Eqs. (2.10) and (4.8a),

$$\Delta F_{II}^{\text{TE}} = \frac{(4\pi\sigma)^3 \alpha}{48\pi^2} \Phi t^{5/2} + O(t^{7/2}), \quad (4.24a)$$

or in SI units with  $\Phi = \zeta(-\frac{3}{2})$ ,

$$\Delta F_{II}^{\text{TE}} = \frac{\sqrt{2}\pi\zeta\left(-\frac{3}{2}\right)a(\sigma^{\text{SI}}/\epsilon_0)^{3/2}}{6\hbar^{3/2}}(k_B T)^{5/2} + \dots \quad (4.24b)$$

Comparison with Eq. (4.4b) shows full agreement with the result for metals ( $\bar{\epsilon} = 1$ ).

It is worth noting that while the next-to-leading temperature correction is of order  $T^{5/2}$ , the term  $\propto T^3$  in Eq. (4.18) dominates it with respect to  $\alpha$ . Thus in the small  $\sigma$  limit the  $T^{5/2}$  dependency becomes all but imperceptible.

## CONCLUSIONS

We have worked out the two leading terms of the temperature correction to the Casimir-Lifshitz free energy at low temperatures between poor conductors obeying a Drude-type dispersion relation. We have assumed that the materials have a small residual conductivity (compared to the reciprocal of the interplate separation) which is finite and constant with respect to temperature and frequency near  $T=0$ .

The calculation for the TM mode complements that of Ref. [19] where the temperature correction for free energy between two dielectrics of zero conductivity was calculated. Both the TE and TM free energy temperature corrections are quadratic to leading order. To the extent of our computations, the TM mode has integer powers of  $T$  beyond the leading, whereas the TE mode has both integer and half-integer powers. The temperature anomaly reviewed in Sec. I occurs when the transition from finite to zero conductivity happens at exactly  $T=0$ , and while the analysis presented here does not resolve the anomaly, it is of interest to know the low-temperature behavior of the free energy in each of the two cases.

Note furthermore that the effects of the static dielectric permittivity  $\bar{\epsilon}$  only enters to order  $T^4$  for the TM mode and order  $T^{7/2}$  for the TE mode. The fact that the coefficient of the term  $T^{7/2}$  appears to depend on  $\bar{\epsilon}$  is noteworthy since only integer powers of  $T$  were reported in Ref. [19], although seeing as we have not calculated the coefficient here it is possible that cancellations occur.

Our calculations are delicate since they rely on the relative smallness of different parameters simultaneously. We have assumed the parameter  $t$  (essentially temperature  $T$  divided by conductivity  $\sigma$ ) small while at the same time letting  $\sigma$  be small compared to the inverse of the separation  $a$ . This is the reason why the leading order temperature corrections in Eq. (3.31) appear to diverge as  $\sigma$  vanishes. On a deeper level these subtleties stem from noncommuting limits in the Lifshitz formalism which are the cause of anomalies such as that reviewed in Sec. I. Another curious property both of the present calculations and those of Geyer, Klimchitskaya, and Mostepanenko [19] is that the free energy corrections of order in  $T$  just beyond what we have considered here appear to

diverge as  $\bar{\epsilon} \rightarrow \infty$ , as indicated for example by Eqs. (3.34) and (4.23). This limit would *a priori* be expected to yield the ideal metal limit. Such phenomena should be addressed in future studies in the effort to achieve full understanding of the low-temperature behavior of the Casimir force and free energy.

The asymptotics of the Lifshitz formula as frequency and temperature approach zero are fraught with inherent subtleties both mathematical and physical. While the method employed herein is highly useful for its simplicity and transparency, it has limitations because the functions involved are not analytic in the limits considered and noninteger powers and logarithms enter. Physically we have assumed herein a model which may represent certain physical systems, but avoids the temperature behavior which leads to the anomaly reviewed in Sec. I. It also neglects effects which may be of importance, such as spatial dispersion, a subject which has been extensively investigated over the years [31–34]. A theoretical effort to attempt to describe the screening effects and dielectric response of the vanishing density of free charges in insulators near zero temperature involving all important physical effects will likely be required in the future and will hopefully provide the resolution of the anomaly for dielectrics.

#### ACKNOWLEDGMENTS

K.A.M.'s research is supported in part by a grant from the US National Science Foundation (Grant No. PHY-0554926) and by a grant from the US Department of Energy (Grant No. DE-FG02-04ER41305). S.A.E. acknowledges the University of Oklahoma for its hospitality while working on this project. We have benefited from discussions and suggestions from Emilio Elizalde, Klaus Kirsten, and Jef Wagner.

#### APPENDIX: BOREL SUMMATION

The Borel sum of the divergent series  $\sum_{n=0}^{\infty} a_n$  exists (Ref. [35], Sec. 8.2) if the function

$$\phi(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} \quad (\text{A1})$$

is convergent for sufficiently small  $x$  and the integral

$$\mathcal{B}(x) = \int_0^{\infty} dt e^{-t} \phi(xt) \quad (\text{A2})$$

exists. Then the Borel sum is  $\sum_{n=0}^{\infty} a_n = \mathcal{B}(1)$ . Consider the quantity  $\Psi$  defined in Eq. (2.11a) and consider the term

$$\tilde{\Psi} = \sum_{n=2}^{\infty} \frac{B_{2n}(2n-4)!}{(2n)!} = \sum_{n=4}^{\infty} \frac{B_n(n-4)!}{n!} \quad (\text{A3})$$

(the latter equality follows from  $B_{2n+1}=0$ ,  $n=1,2,\dots$ ). Letting  $a_{n-4}=B_n(n-4)!/n!$  we obtain

$$\phi(x) = \sum_{n=0}^{\infty} \frac{B_{n+4}x^n}{(n+4)!} = \frac{1}{x^4} \left[ \frac{x}{e^x-1} - 1 + \frac{x}{2} - \frac{x^2}{12} \right], \quad (\text{A4})$$

where the identity  $\sum_{n=0}^{\infty} B_n x^n / n! = x / (e^x - 1)$  was used. The Borel sum

$$\tilde{\Psi} = \mathcal{B}(1) = \int_0^{\infty} \frac{dt e^{-t}}{t^4} \left[ \frac{t}{e^t-1} - 1 + \frac{t}{2} - \frac{t^2}{12} \right] \quad (\text{A5})$$

is now possible to evaluate analytically (the divergence in the lower limit is illusory because the expression in brackets is of order  $t^4$ ) to find the desired value as  $\Psi = 1/36 - 2\tilde{\Psi} = \zeta(3)/(4\pi^2)$ . A similar numerical procedure using Eqs. (A1) and (A2) will give the value of  $\Phi$ , which as noted in the text has a well-known asymptotic expression.

An alternative and equivalent approach which is most often simpler is to sum each term of the expansion of  $g(m)$  in Eq. (2.5) directly (disregarding the zero temperature contribution) and obtain finite values of the terms of the temperature expansion by means of  $\zeta$  regularization [36]. Such a procedure immediately yields  $\Phi = \zeta(-\frac{3}{2})$  by the definition of the  $\zeta$  function as the analytic continuation of  $\zeta(s) = \sum_{m=1}^{\infty} m^{-s}$ . Likewise the value of  $\Psi$  can easily be found by comparison with the asymptotic series expansion of the derivative of  $\zeta(s)$

$$\zeta'(s) = - \sum_{n=1}^{\infty} n^{-s} \ln n, \quad (\text{A6})$$

whereby  $\Psi = -\zeta'(-2) = \zeta(3)/4\pi^2$ . The same reasoning also yields the coefficient of the  $c_1$  term of Eq. (2.10) directly as  $\zeta(-1) = -1/12$ . We thank Emilio Elizalde for alerting us to this point.

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